SYMPLECTIC QUOTIENTS: MOMENT MAPS, SYMPLECTIC REDUCTION AND THE MARSDEN-WEINSTEIN-MEYER THEOREM

VICTORIA HOSKINS

1. Construction of group quotients in differential geometry

Let X be a smooth manifold and K be a Lie group; then an action of K on X is an action $\sigma : K \times X \to X$ which is smooth map of manifolds such that $\sigma_k : X \to X$ is a diffeomorphism for all $k \in K$. Let $K \cdot x$ denote the orbit of $x \in X$ and K_x denote the stabiliser of x. The topological quotient is the orbit space

$$X/K = \{K \cdot x : x \in X\},$$

which is given the quotient topology, so that the morphism $\pi : X \to X/K$ is continuous. In general, this topological quotient need not be a manifold.

Example 1.1.

(1) Let S^1 act on S^2 by rotation so that the North and South pole are the only fixed points of this action; then the topological quotient is homeomorphic to the closed interval $[0, 1]$, which is not a manifold, as it has corners. In this example, the boundary points arise as the action is not free.
(2) Let $\mathbb{R}$ act on $T^2 = S^1 \times S^1$ by translation by an irrational slope ω_1/ω_2. More precisely, if we consider T^2 as obtained from the square $[0, 1]^2$ by gluing opposite edges, then the lift of this action to $[0, 1]^2$ is given by

$$t \cdot (a, b) = (a + \omega_1 t, b + \omega_2 t).$$

In this example, the orbit space is not Hausdorff due to the presence of dense orbits. This is a phenomena of non-proper actions.

Remark 1.2.

(1) If K acts properly on X (that is, the graph of the action $K \times X \to X \times X$ given by $(k, x) \mapsto (k \cdot x, x)$ is proper), then all orbits are closed and the quotient X/K is Hausdorff.
(2) If K is compact, then any action of K on X is proper.

Theorem 1.3. *Let K be a Lie group acting freely and properly on a smooth manifold X; then there is a unique structure of a smooth manifold on X/K such that $\pi : X \to X/K$ is a submersion. Furthermore, $\pi : X \to X/K$ is a principal K-bundle.*

Proof. We give an outline of the main steps in the proof.

Step 1: each orbit $K \cdot x$ is a closed embedded submanifold diffeomorphic to K. Consider the action map $\sigma_x : K \to X$ given by $k \mapsto k \cdot x$; this is a smooth (as σ is smooth), proper (as K is compact), injective (as the action is free) morphism with image $K \cdot x$. We claim that σ_x is an immersion. Up to the action of K, it suffices to check that $d_e\sigma_x : \mathfrak{k} \to T_xX$ is injective, where $\mathfrak{k}$ denotes the Lie algebra of K. We see that $d_e\sigma_x(A) = 0$ for $A \in \mathfrak{k}$ if and only if the infinitesimal action of A on x is trivial, but, as the action is free, this holds if and only if $A = 0$.

Step 2: the Slice theorem. For a free action of K on X, a slice of the action at $x \in X$ is an open neighbourhood $S \subset X$ of x such that the action $K \times S \to K \cdot S$ gives a K-equivariant diffeomorphism. In our set up of a free and proper action, we can use the Slice theorem of Palais to construct slices of the action at every point in X and, in Step 3, we use these slice to provide charts on X/K.

To construct a slice of the action at $x \in X$, we choose a decomposition

$$T_xX = T_x(K \cdot x) \oplus W$$

such that W is normal to the orbit. Following Step 1, we produce a transverse section S to the orbit $K \cdot x$ by taking a sufficiently small ball $D \subset W$ such S is the image of D under the diffeomorphism

$$\Phi : \{\text{neighbourhoods of } 0 \in T_xX\} \longrightarrow \{\text{neighbourhoods of } x \in X\}.$$

For $\epsilon > 0$, we let $S_\epsilon := S \cap \Phi(B_\epsilon(0))$. The Slice theorem states that for ϵ sufficiently small, the map $\eta_\epsilon : K \times S_\epsilon \to X$ given by $(k, s) \mapsto k \cdot s$ is a diffeomorphism onto a K-invariant neighbourhood of $K \cdot x$. For any $\epsilon > 0$, one verifies that $d_{(e,x)}\eta_\epsilon$ is bijective; that is, η_ϵ is a local diffeomorphism at (e, x) and, using the K-action, we deduce that $d\eta_\epsilon$ is a local diffeomorphism at (k, x) for all $k \in K$. We claim that for ϵ sufficiently small, η_ϵ is injective. To prove this we argue by contradiction: otherwise there are sequences $x_n \to x$ and $x'_n \to x$ in X and $g_n, h_n \in K$ such that $(g_n, x_n) \neq (h_n, x'_n)$ and

$$g_n \cdot x_n = h_n \cdot x'_n.$$

Let $k_n := h_n^{-1}g_n$; then $k_n \cdot x_n = x'_n \to x$. Consider the graph of the action $\Gamma : K \times X \to X \times X$; then $\Gamma(k_n, x_n) = (k_n \cdot x_n, x_n) = (x'_n, x_n)$, which converges to (x, x). As the action is proper, there is a convergent subsequence $k_{n_j} \to k$. However, such a convergent subsequence contradicts the fact that η_ϵ is a local diffeomorphism at (k, x). Hence, for ϵ sufficiently small, η_ϵ is injective, and provides a diffeomorphism from a neighbourhood $K \times S_\epsilon$ of $K \times \{x\}$ in $K \times X$ onto a neighbourhood U of $\eta(K \times \{x\}) = K \cdot x$ in X. This completes the proof of the Slice theorem.

Step 3: The construction of charts via the slice theorem. Let $\pi : X \to X/K$ and $p = \pi(x)$ for some $x \in X$. We use the slice theorem to provide a chart at $p \in X/K$. Let S be a slice at $x \in X$; then S is a smooth open neighbourhood of $x \in X$ and we have a diffeomorphism $K \times S \cong U := K \cdot S$. Hence,

$$\pi(U) = U/K \cong (K \times S)/K \cong S$$

is an open neighbourhood of $p = \pi(x) \in X/K$ and we can use the smooth structure on S to define a smooth structure locally on X/K. We leave it as an exercise to check that the transition functions and π are smooth. By construction of these charts, we see that $\pi : X \to X/K$ is a principal K-bundle. □

2. Actions in symplectic geometry

Let (X, ω) be a symplectic manifold; that is, X is a (real) smooth manifold and ω is a closed non-degenerate 2-form on X, called the symplectic form. We can think of ω as a family of skew-symmetric non-degenerate bilinear forms

$$\omega_x : T_xX \times T_xX \to \mathbb{R}.$$

The non-degeneracy of ω implies that $\dim_{\mathbb{R}} X$ is even and ω induces isomorphisms $T_xX \cong T_xX^*$, which vary smoothly with $x \in X$. In particular, the symplectic form determines an isomorphism $TX \cong T^*X$.

Definition 2.1. Let K be a lie group and $\sigma : K \times X \to X$ be a smooth action. We say the action is *symplectic* if K acts by symplectomorphisms; that is, $\sigma_k^*\omega = \omega$ for all $k \in K$.

Hence, for a symplectic action, the action $K \to \text{Diff}(X)$ factors through the subgroup of symplectomorphisms $\text{Sympl}(X, \omega)$.

Our goal is to provide a method for constructing quotients of such actions in symplectic geometry; in the sense, that the quotient is also a symplectic manifold. Even if a symplectic K-action on (X, ω) admits a smooth quotient X/K, this quotient may not be symplectic purely for dimension reasons (for example, if K has odd dimension, then so does X/K and so the quotient cannot be symplectic). Instead, we will construct a quotient with expected dimension $\dim X - 2\dim K$, by using a moment map for the action to construct a symplectic reduction. The moment map for the action can be thought of as a lift of an infinitesimal version of the action.

2.1. **Infinitesimal action.** The Lie algebra $\mathfrak{k} = T_e K$ is an infinitesimal version of the Lie group K. We can construct an infinitesimal version of the action by associating to each $A \in \mathfrak{k}$, a vector field A_X on X as follows: at $x \in X$, we have

$$A_{X,x} := \frac{d}{dt} \exp(tA) \cdot x|_{t=0} \in T_x X.$$

We denote the (infinite dimensional) Lie algebra of smooth vector fields on X by $\mathrm{Vect}(X) := \Gamma(TX)$, where the Lie bracket $[-,-]$ on $\mathrm{Vect}(X)$ is given by the commutator. The *infinitesimal action*

$$\sigma_{\mathrm{inf}} : \mathfrak{k} \to \mathrm{Vect}(X), \quad A \mapsto A_X$$

is a Lie algebra anti-homomorphism (the fact that this is an anti-homomorphism is explained by the sign which occurs when going between left and right actions: the Lie bracket on $\mathfrak{k}$ is defined using left-invariant vector fields, which are generating vector fields for the right action of the group on itself, whereas the action of K on X is a left action).

The symplectic form ω determines an isomorphism $TX \cong T^*X$ and, on taking global sections, we obtain an isomorphism

$$\mathrm{Vect}(X) \cong \Omega^1(X) := \Gamma(T^*X).$$

The exterior derivative $d : \mathcal{C}^\infty(X) \to \Omega^1(X)$ together with this isomorphism determine a map

$$\Phi_\omega : \mathcal{C}^\infty(X) \to \Omega^1(X) \cong \mathrm{Vect}(X), \quad f \mapsto Y_f.$$

We can define a Lie algebra structure on $\mathcal{C}^\infty(X)$ using ω by

$$\{f, g\} := \omega(Y_f, Y_g)$$

where $Y_f = \Phi_\omega(f)$. Then Φ_ω is a Lie algebra anti-homomorphism; that is,

$$\Phi_\omega(\{f, g\}) = -[Y_f, Y_g].$$

Definition 2.2. A vector field on a symplectic manifold (X, ω) is *Hamiltonian* if it lies in the image of Φ_ω (that is, the corresponding 1-form is exact). A vector field is *symplectic* if the corresponding 1-form is closed.

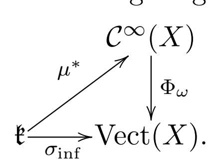
As exact forms are closed, every Hamiltonian vector field is a symplectic vector field.

Definition 2.3. Let K be a Lie group acting on a symplectic manifold (X, ω). The action is

i) *infinitesimally symplectic* if A_X is a symplectic vector field for all $A \in \mathfrak{k}$.
ii) *weakly Hamiltonian* if A_X is a Hamiltonian vector field for all $A \in \mathfrak{k}$.

For a weakly Hamiltonian action, the infinitesimal action $\mathfrak{k} \to \mathrm{Vect}(X)$ can be pointwise lifted to a map $\mathfrak{k} \to \mathcal{C}^\infty(X)$, because a Hamiltonian vector field corresponds (under ω) to the exterior derivative of a smooth function on X. In general, this lift is non-unique as there may be several smooth functions with the same exterior derivative.

Definition 2.4. A symplectic action of a Lie group K on (X, ω) is *Hamiltonian* if the infinitesimal action $\sigma_{\mathrm{inf}} : \mathfrak{k} \to \mathrm{Vect}(X)$ can be lifted to a Lie algebra homomorphism $\mu^* : \mathfrak{k} \to \mathcal{C}^\infty(X)$, called the *comoment map*, such that the following diagram commutes



2.2. **Moment map.** Hamiltonian actions can also be described by using a moment (or momentum) map, which one should think of as dual to a comoment map.

Definition 2.5. A smooth map $\mu : X \to \mathfrak{k}^*$ is called a *moment map* if it is K-equivariant with respect to the action of the given K on X and the coadjoint action of K on $\mathfrak{k}^*$, and μ satisfies the following infinitesimal lifting property:

$$d_x\mu(\zeta) \cdot A = \omega_x(A_{X,x}, \zeta) \tag{1}$$

for all $x \in X$, $\zeta \in T_x X$ and $A \in \mathfrak{k}$.

Remark 2.6.

(1) For $\mu : X \to \mathfrak{k}^*$ and $A \in \mathfrak{k}^*$, if we let $\mu^A : X \to \mathbb{R}$ denote the function $x \mapsto \mu(x) \cdot A$, then the infinitesimal lifting property can be stated as $d\mu^A = \iota_{A_X}\omega$ for all $A \in \mathfrak{k}^*$, where $\iota_A\omega$ denotes the 1-form obtained by contracting the 2-form ω with the vector field A_X.

(2) The momentum map first arose in classical mechanics; we give some simple examples of moment maps arising from physical actions in Example 2.8 below. For a detailed description of the moment map from the viewpoint of classical mechanics, see the notes of Butterfield [1].

(3) There is still no consistent sign convention for the moment map and so often a minus sign may appear in the condition (1) used to define a moment map.

For K connected, we claim that there is a bijective correspondence

$$\{\mu : X \to \mathfrak{k}^* \text{ moment maps}\} \longleftrightarrow \{\mu^* : \mathfrak{k} \to \mathcal{C}^\infty(X) \text{ comoment maps}\}$$

given by $\mu \mapsto \mu^*$ where $\mu^*(A)(x) := \mu(x) \cdot A$. It is clear that the infinitesimal lifting property of μ given by (1) corresponds to the factorisation property $\varphi_\omega \circ \mu^* = \sigma_{\text{inf}}$. Let us explain how K-equivariance of μ corresponds to the property that μ^* is a Lie algebra homomorphism. Suppose we have a moment map μ; then by K-equivariance of μ, we have

$$\mu(k \cdot x) \cdot A = (\mathrm{Ad}_k^* \mu(x)) \cdot A = \mu(x) \cdot \mathrm{Ad}_{k^{-1}}(A)$$

for all $k \in K$, $x \in X$ and $A \in \mathfrak{k}$. For $A, B \in \mathfrak{k}$, we then have

$$\begin{aligned} 0 &= \frac{d}{dt}\, \mu(\exp(tB) \cdot x) \cdot A - \mu(x) \cdot \mathrm{Ad}_{\exp(-tB)}(A) \,|_{t=0} \\ &= d_x\mu(B_{X,x}) \cdot A - \mu(x) \cdot [-B, A] \\ &= \omega_x(A_{X,x}, B_{X,x}) - \mu(x) \cdot [A, B] \\ &= \{\mu^*(A), \mu^*(B)\}(x) - \mu^*([A, B])(x), \end{aligned}$$

which proves that μ^* is an Lie algebra homomorphism. Conversely, if we know that μ^* is a Lie algebra homomorphism and we want to prove that μ is K-equivariant, it suffices to prove that for all $x \in X$ and $A \in \mathfrak{k}^*$ that the map $\varphi_{x,A} : K \to \mathbb{R}$ given by $\varphi_{x,A}(k) := \mu(k \cdot x) \cdot \mathrm{Ad}_k(A)$ is constant. As K is connected, it suffices to prove that the derivative of $\varphi_{x,A}$ is trivial and, by using the K-action, we can reduce to checking only $d_e\varphi_{x,A} = 0$. We see that

$$\begin{aligned} d_e\varphi_{x,A}(B) &= \frac{d}{dt}\mu(\exp(tB) \cdot x) \cdot \mathrm{Ad}_{\exp(tB)}(A)|_{t=0} \\ &= d_x\mu(B_{X,x}) \cdot A + \mu(x) \cdot [B, A] \\ &= \{\mu^*(A), \mu^*(B)\}(x) - \mu^*([A, B])(x) = 0, \end{aligned}$$

as μ^* is a Lie algebra homomorphism, which completes the proof.

Remark 2.7. The moment map is not necessarily unique (see Example 2.9 below), although for certain groups we will see that it is unique (cf. part (1) of Example 2.10). The existence of a moment map can be characterised in terms of an extension of ω to a equivariant 2-form by work of Atiyah and Bott; see §2.4 below.

2.3. **Examples of moment maps.** We start with some examples which highlight the connection between the moment map and classical mechanics.

Example 2.8.

(1) Consider the natural action of SO(3) on the sphere $S^2 := \{x^2+y^2+z^2 = 1\} \subset \mathbb{R}^3$, where S^2 has symplectic structure $\omega = d\theta \wedge dz$, for the cylindrical coordinates r, θ, z on $\mathbb{R}^3$. We can identify $\mathfrak{so}(3) \cong \mathbb{R}^3$ by sending the infinitesimal rotation at the standard basis vector e_j to e_j. Then the moment map for this action is the inclusion $S^2 \hookrightarrow \mathbb{R}^3 \cong \mathfrak{so}(3)^*$. To see this, consider the S^1-action on S^2 given by rotation around the z-axis. Then, if we identify $T_1S^1 \cong \mathbb{R}$, the moment map is given by the projection $(x, y, z) \to z$, as

$$\iota_{\frac{\partial}{\partial\theta}}\omega = dz.$$

We note that this map is S^1-equivariant, as it is S^1-invariant and the coadjoint action of S^1 is trivial. Similarly, one can verify that the S^1-action given by rotation around the x-axis (resp. y-axis) is the projection onto the corresponding axis.

(2) If one considers the lift of the natural action of SO(3) on $\mathbb{R}^3$ to its cotangent bundle $T^*\mathbb{R}^3 \cong \mathbb{R}^3 \times \mathbb{R}^3$ given by the diagonal action, then this action is symplectic for the Liouville form ω on $T^*\mathbb{R}^3$. Furthermore, the moment map $\mu : T^*\mathbb{R}^3 \to \mathbb{R}^3$ is given by the cross product $\mu(p, q) = p \times q$. If we view p as a position vector and q as a momentum vector, then the cross product $p \times q$ gives the angular momentum about the origin.

The simplest example is actions on symplectic vector spaces given by representations.

Example 2.9. Let $K = \mathrm{U}(n)$ be the unitary group of $n \times n$ matrices and consider its standard representation on $\mathbb{C}^n$. The infinitesimal action is given by

$$A_{X,x} = \frac{d}{dt}\exp(tA) \cdot x|_{t=0} = Ax$$

for $x \in \mathbb{C}^n$ and A a skew-Hermitian matrix in $\mathfrak{u}(n)$. Let $H : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ denote the standard Hermitian inner product on $\mathbb{C}^n$: if we write $z \in \mathbb{C}^n$ as a row vector $(z_1, \dots z_n)$, then

$$H(z, v) = z\overline{v}^t = \overline{v}z^t = \mathrm{Tr}(z^t\overline{v}) = \mathrm{Tr}(\overline{v}^t z)$$

where v^t denotes the transpose of a matrix. We take the symplectic form $\omega : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{R}$ equal given by the imaginary part of this Hermitian inner product:

$$\omega(z, v) = \frac{1}{2i}\left(H(z, v) - \overline{H(z, v)}\right) = \frac{1}{2i}\left(H(z, v) - H(v, z)\right).$$

The Hermitian inner product is $\mathrm{U}(n)$-invariant (that is, $H(Az, Av) = H(z, v)$ for all unitary matrices $A \in \mathrm{U}(n)$) and so it follows that the action of $\mathrm{U}(n)$ on $(\mathbb{C}^n, \omega)$ is symplectic. In fact, this action is Hamiltonian and there is a canonical choice of moment map $\mu : \mathbb{C}^n \to \mathfrak{u}(n)^*$ which is defined by

$$\mu(z) \cdot A = \frac{1}{2}\omega(Az, z) = \frac{1}{2i}H(Az, z)$$

for $A \in \mathfrak{u}(n)$. The second equality follows from the fact that for $A \in \mathfrak{u}(n)$:

$$H(Az, v) + H(z, Av) = \mathrm{Tr}(z^t A^t \overline{v}) + \mathrm{Tr}(z^t \overline{A}\overline{v}) = \mathrm{Tr}(z^t(A^t + \overline{A})\overline{v}) = \mathrm{Tr}(0) = 0 \tag{2}$$

for all $z, v \in \mathbb{C}^n$ and $A \in \mathfrak{u}(n)$. We shall now carefully check that this is a moment map. First, μ is $\mathrm{U}(n)$-equivariant:

$$\mu(k \cdot z) \cdot A = \frac{1}{2}\omega(Ak \cdot z, k \cdot z) = \frac{1}{2}\omega(k^{-1}Ak \cdot z, \cdot z) = \mu(z) \cdot k^{-1}Ak = \mathrm{Ad}^*_k\mu(x) \cdot A$$

where $A \in \mathfrak{u}(n)$, $z \in \mathbb{C}^n$ and $k \in \mathrm{U}(n)$. To verify the infinitesimal lifting condition (1), we may identify $T_z\mathbb{C}^n \cong \mathbb{C}^n$ and then this condition becomes

$$d_z\mu(v) \cdot A = \omega(Az, v)$$

for $v \in \mathbb{C}^n \cong T_z\mathbb{C}^n$ and $A \in \mathfrak{u}(n)$. We have

$$\begin{aligned}
d_z\mu(v) \cdot A &:= \frac{d}{dt}\mu(z + tv) \cdot A|_{t=0} \\
&= \frac{1}{2}\frac{d}{dt}\omega(A(z + tv), z + tv)|_{t=0} = \frac{1}{2}\left[\omega(Az, v) + \omega(Av, z)\right] \\
&= \frac{1}{4i}\left[H(Az, v) - H(v, Az) + H(Av, z) - H(z, Av)\right] \\
&= \frac{1}{2i}\left[H(Az, v) - H(v, Az)\right] =: \omega(Az, v)
\end{aligned}$$

where the first equality on the final line follows from the relation given at (2).

The moment map for this symplectic action is not unique; although it is unique up to addition by an element η of $\mathfrak{u}(n)^*$ which is fixed by the coadjoint action $\mathrm{U}(n) \to \mathrm{GL}(\mathfrak{u}(n))$ (we call such η a central element). Every character of $\mathrm{U}(n)$ is a power of the determinant $\det : \mathrm{U}(n) \to \mathrm{S}^1$, whose derivative is given by the trace $\mathrm{Tr} : \mathfrak{u}(n) \to \mathrm{Lie}\,\mathrm{S}^1 \cong 2\pi i\mathbb{R}$. Hence, such a central element

η of $\mathfrak{u}(n)^*$ must be equal to $ci\mathrm{Tr} \in \mathfrak{u}(n)^*$, for some constant $c \in \mathbb{R}$, and we can use such an element to define a shifted moment map

$$\mu_\eta(z) \cdot A = \frac{1}{2i} H(Az, z) + \eta \cdot A.$$

Example 2.10. If K is a Lie group which acts on $\mathbb{C}^n$ by a faithful representation $\rho : K \to \mathrm{U}(n)$, then we can write down the moment map μ_K for the action of K on $\mathbb{C}^n$ using ρ and the moment map $\mu_{\mathrm{U}(n)} : \mathbb{C}^n \to \mathfrak{u}(n)^*$ for the $\mathrm{U}(n)$-action on $\mathbb{C}^n$ constructed in Example 2.9. The moment map for the action of K is given by

$$\mu_K = \rho^* \mu_{\mathrm{U}(n)}$$

where $\rho^* : \mathfrak{u}(n)^* \to \mathfrak{k}^*$ is the dual to the inclusion $\rho : \mathfrak{k} \to \mathfrak{u}(n)$. We consider the following special cases:

(1) If $K = \mathrm{SU}(n)$ acts on $\mathbb{C}^n$ by the standard inclusion into $\mathrm{U}(n)$, then its moment map is given by

$$\mu(z) \cdot A = \frac{1}{2}\omega(Az, z) = \frac{1}{2i} H(Az, z)$$

for $A \in \mathfrak{su}(n)$. However, there are no non-zero central elements of $\mathfrak{su}(n)$ which we can use to shift the moment map by and so this moment map is unique.

(2) If $K = (S^1)^n$ acts on $\mathbb{C}^n$ via the representation

$$(t_1, \dots, t_n) \mapsto \mathrm{diag}(t_1, \dots, t_n),$$

then the moment map is given by

$$\mu(z) \cdot (A_1, \dots, A_n) = \frac{1}{2}\omega(\mathrm{diag}(A_1, \dots, A_n)z, z)$$

for $A_k \in \mathrm{Lie}S^1 \cong 2\pi i\mathbb{R}$. If we write $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and $A_k = 2ia_k$ for real numbers a_k then

$$\mu(z) \cdot (2ia_1, \dots, 2ia_n) = \sum_{k=1}^{n} a_k |z_k|^2.$$

As the group $K = (S^1)^n$ is commutative, every element in central and so it follows that all elements in $\mathfrak{k} \cong (2\pi i\mathbb{R})^n$ are central. We can shift the standard moment map by any n-tuple $(c_1, \dots, c_n)$ of real numbers to get

$$\mu(z) \cdot (2ia_1, \dots, 2ia_n) = \sum_{k=1}^{n} (a_k |z_k|^2 + c_k).$$

(3) We may also consider $K = (S^1)^n$ acting on $\mathbb{C}^n$ via the representation

$$(t_1, \dots, t_n) \mapsto \mathrm{diag}(t_1^{r_1}, \dots, t_n^{r_n})$$

for integers r_k. In this case the moment map (shifted by real numbers c_i) is given by

$$\mu(z) \cdot (2ia_1, \dots, 2ia_n) = \sum_{k=1}^{n} (a_k r_k |z_k|^2 + c_k).$$

Exercise 2.11. Consider the action of $K = U(m)$ on the space of $l \times m$-matrices over the complex numbers $M_{l\times m}(\mathbb{C}) \cong \mathbb{C}^{lm}$ given by $k \cdot M = Mk^{-1}$ where we take the natural symplectic structure given by the imaginary part of the standard Hermitian inner product on $\mathbb{C}^{lm}$. Then if $M \in M_{l\times m}$ and $A \in \mathfrak{u}(m)$ show

$$\mu(M) \cdot A = \frac{i}{2} \mathrm{Tr}(MAM^*)$$

is a moment map for this action.

So far the only examples of moment maps that we have seen are for affine spaces. A more interesting example is complex projective space $\mathbb{P}^n$ with its Fubini–Study symplectic form ω_{FS} (or a smooth closed subvariety with the induced form). We consider a linear unitary action of K on $\mathbb{P}^n$ (that is, K acts via a faithful representation $K \to \mathrm{U}(n+1)$). In fact, the moment map $\mu_K : \mathbb{P}^n \to \mathfrak{k}^*$ for the K-action is the composition of the moment map $\mu : \mathbb{P}^n \to \mathfrak{u}(n+1)^*$ for the $U(n+1)$-action followed by the projection $\mathfrak{u}(n+1)^* \to \mathfrak{k}^*$.

Example 2.12. Let $\mathrm{U}(n+1)$ act on complex projective space $\mathbb{P}^n$ by acting on its affine cone $\mathbb{C}^{n+1}$ by its standard representation. The symplectic form on $\mathbb{P}^n$ is the Fubini-Study form ω_{FS} constructed from the standard Hermitian inner product H on $\mathbb{C}^{n+1}$. It is easy to see this from is $\mathrm{U}(n+1)$-invariant (that is, the action is symplectic). As the Fubini-Study form is preserved by the action of $\mathrm{U}(n+1)$ and the unitary group is compact and connected, the action is infinitesimally symplectic so that $d\iota_{A_X}\omega_{\text{FS}} = 0$. Moreover, as $H^1(\mathbb{P}^n) = 0$, every closed 1-form is exact and so the action is weakly Hamiltonian. It turns out that this action is Hamiltonian and we can write down a moment map explicitly as follows. For $p = (p_0, \dots, p_n) \in \mathbb{C}^{n+1} - \{0\}$, we claim that

$$\mu([p]) \cdot A = \frac{\mathrm{Tr} p^* A p}{2i||p||^2}$$

defines a moment map where $[p] \in \mathbb{P}^n$ and $A \in \mathfrak{u}(n+1)$ and p^* denotes the complex conjugate transpose. We leave the $\mathrm{U}(n+1)$-equivariance of μ as an exercise for the reader to check. As the action of $\mathrm{U}(n+1)$ on $\mathbb{P}^n$ is transitive, we need only verify the condition (1) at a single point $[p] = [1 : 0 : \cdots : 0] \in \mathbb{P}^n$. We can identify $T_{[p]}\mathbb{P}^n$ with the orthogonal space to p in $\mathbb{C}^{n+1}$ with respect to the standard Hermitian product on $\mathbb{C}^{n+1}$:

$$T_{[p]}\mathbb{P}^n = T_p S^{2n+1}/T_p(S^1 \cdot p) \cong \{(0, z_1, \dots, z_n) \in \mathbb{C}^{n+1}\} \cong \mathbb{C}^n.$$

With respect to the coordinates $(z_1, \dots, z_n) \mapsto [1 : z_1 : \cdots : z_n]$ at $[p]$, the Fubini-Study form can be expressed locally as

$$\omega_{\text{FS},[p]} = \frac{1}{2i} \sum_{k=1}^{n} dz_k \wedge d\bar{z}_k;$$

that is, for $v, w \in T_{[p]}\mathbb{P}^n$, we have

$$\omega_{\text{FS},[p]}(v, w) = \mathrm{Im} H(v, w) = \frac{1}{2i}[H(v, w) - H(w, v)].$$

Let $v \in T_{[p]}\mathbb{P}^n$ and $A \in \mathfrak{u}(n+1)$; then

$$\begin{aligned} d_{[p]}\mu(v) \cdot A &:= \frac{d}{dt}\mu(p + tv) \cdot A|_{t=0} = \frac{1}{2i}\frac{d}{dt}\frac{\mathrm{Tr}((p+tv)^* A(p+tv))}{||p+tv||^2}|_{t=0} \\ &= \frac{1}{2i}\frac{\mathrm{Tr}(v^*Ap + p^*Av)||p||^2 - (pv^* + vp^*)\mathrm{Tr}(p^*Ap)}{||p||^4} \\ &\overset{(a)}{=} \frac{1}{2i}[H(Ap, v) - H(v, Ap)] \\ &= \omega_{\text{FS},[p]}(Ap, v) \end{aligned}$$

where (a) follows as $H(p, v) = p^*v = 0$ for $v \in T_{[p]}\mathbb{P}^n$.

Exercise 2.13. Let K be a compact and connected Lie group; then a coadjoint orbit $\mathcal{O} \subset \mathfrak{k}^*$ for the action of K on $\mathfrak{k}^*$ has a symplectic form ω given by the Kostant-Kirillov symplectic structure. Describe the infinitesimal action for the natural action of K on $\mathcal{O}$ and show that the inclusion $\mu : \mathcal{O} \hookrightarrow \mathfrak{k}^*$ is a moment map for this action.

2.4. **Moment maps and infinitesimal linearisations.** Given an action of a Lie group K on a smooth manifold X, Cartan introduced a notion of K-equivariant forms which are K-invariant polynomials on $\mathfrak{k}$ with coefficients in $\Omega^*(X)$. More precisely, let

$$\Omega^l_K(X) := \bigoplus_{2i+j=l} \left(\mathrm{Sym}^i(\mathfrak{k}^*) \otimes \Omega^j(X)\right)^K,$$

where $(\mathrm{Sym}^i(\mathfrak{k}^*)\otimes\Omega^j(X))^K$ is the space of K-equivariant polynomial maps on $\mathfrak{k}$ of homogeneous degree i with coefficients in $\Omega^j(X)$. There is an equivariant derivative

$$d_K : \Omega^l_k(X) \to \Omega^{l+1}_K(X), \quad (d_K(\beta))(A) := (d - \iota_{A_X})(\beta(A)),$$

where $A \in \mathfrak{k}$ and $\iota_{A_X} : \Omega^j(X) \to \Omega^{j-1}(X)$ denotes the contraction with the vector field A_X given by the infinitesimal action of A.

Now suppose that K acts symplectically on (X, ω). Then Atiyah and Bott characterise moment maps for the K-action on X as K-equivariantly closed extensions of the symplectic form ω to $\Omega^2_K(X)$ in the following way. Since we have

$$\Omega^2_K(X) = \Omega^2(X)^K \oplus \mathrm{Hom}(\mathfrak{k}, \Omega^0(X))^K \cong \Omega^2(X)^K \oplus \mathrm{Map}_K(X, \mathfrak{k}^*),$$

any extension of $\omega \in \Omega^2(X)^K$ to a K-equivariant 2-form is of the form $\omega+\mu^*$ for a K-equivariant map $\mu : X \to \mathfrak{k}^*$. This extension is K-equivariantly closed if and only if

$$0 = d_K(\omega - \mu^*)(A) = (d\omega, -\iota_{A_X}\omega + d\mu^A);$$

that is, if and only if μ is a moment map.

Given an Hermitian line bundle $\pi : L \to X$, the group $U(1)$ acts on the associated unit circle bundle $\pi_1 : L_1 \to X$. Suppose that we have a $U(1)$-invariant connection 1-form θ on L_1 with curvature $2\pi i\omega \in \Omega^2(X)$ (that is, $\pi_1^*\omega = -d\theta$ to fix our sign conventions). Then any unitary automorphism of L preserving θ descends to an automorphism of X preserving ω; that is, a symplectomorphism of (X, ω). Furthermore, there is a short exact sequence

$$1 \to U(1) \to \mathrm{Aut}(L_1, \theta) \to \mathrm{Sympl}(X, \omega) \to 1.$$

The K-action on X determines a morphism $K \to \mathrm{Sympl}(X, \omega)$. A *linearisation* of the K-action on X in the Hermitian line bundle L is a lift $K \to \mathrm{Aut}(L_1, \theta)$ and an *infinitesimal action* is a lift $\mathfrak{k} \to \mathrm{Vect}(L_1)^{U(1)}$ of the infinitesimal action $\mathfrak{k} \to \mathrm{Vect}(X)$. We claim that there is a bijection

$$\{\mu : X \to \mathfrak{k}^* \text{ moment maps}\} \longleftrightarrow \{\phi : \mathfrak{k} \to \mathrm{Vect}(L_1)^{U(1)} \text{ infinitesimal linearisations}\},$$

such that $\mu^A : X \to \mathbb{R}$ corresponds to $\phi^A := \phi(A) \in \mathrm{Vect}(L_1)^{U(1)}$ for $A \in \mathfrak{k}$. More precisely, if $l \in \pi_1^{-1}(x)$, then μ^A and ϕ^A are related by the formula

$$\mu^A(x) = \theta_l(\phi^A_l),$$

where $\theta_l \in T^*_l L_1$ and $\phi^A_l \in T_l L_1$. We can verify that map μ associated to an infinitesimal linearisation ϕ gives an infinitesimal lift of the action:

$$\pi_1^* d\mu^A = d(\theta(\phi^A)) = d\iota_{\phi^A}\theta = (\mathcal{L}_{\phi^A} - \iota_{\phi^A} d)\theta = \mathcal{L}_{\phi^A}\theta + \iota_{\phi^A}\pi_1^*\omega = \pi_1^*\iota_{A_X}\omega,$$

where $\iota_{\phi^A}\pi_1^* = \pi_1^*\iota_{A_X}$ as ϕ is a lift of the infinitesimal action. Hence, we can conclude that $d\mu^A = \iota_{A_X}\omega$. The equivariance of μ follows from the fact that α is $U(1)$-invariant.

3. Symplectic quotients

Given a symplectic action of a Lie group K on a symplectic manifold (X, ω), we can ask whether a quotient exists in the category of symplectic manifolds. The topological quotient always exists, but it may not be a manifold; for example, if the action is not free or proper. Even if the action is free and proper, the resulting quotient manifold may have odd dimension and so will not admit a symplectic form. Hence, the topological quotient X/K does not in general provide a suitable quotient in symplectic geometry.

In this section we define the symplectic reduction of a Hamiltonian action as a quotient of a level set of the moment map. We explain that this symplectic reduction inherits a unique symplectic form by a result of Marsden and Weinstein [5] and Meyer [5]. Finally, we prove that the symplectic reduction satisfied a universal property amongst all symplectic quotients. For further expository reading, see [2], [5] and [9].

3.1. **Properties of moment maps.** Suppose K is a Lie group acting on a symplectic manifold (X, ω). For $x \in X$, we let $K \cdot x$ denote the orbit of x and we let K_x denote the stabiliser of x. We note that $\mathfrak{k}_x := \{A \in \mathfrak{k} : A_{X,x} = 0\}$ is the Lie algebra of K_x.

If the action is Hamiltonian, then there is an associated moment map $\mu : X \to \mathfrak{k}^*$ and one can naturally ask what other properties of the action are encoded by the moment map.

Lemma 3.1. *Let K be a Lie group with a Hamiltonian action on a symplectic manifold (X, ω) with associated moment map $\mu : X \to \mathfrak{k}^*$. For $x \in X$, we have*

i) $\ker d_x\mu = (T_x(K \cdot x))^{\omega_x} := \{\zeta \in T_xX : \omega_x(\eta, \zeta) = 0 \ \forall \, \eta \in T_x(K \cdot x)\}$*;*
ii) $\operatorname{Im} d_x\mu = \operatorname{Ann}\mathfrak{k}_x := \{\eta \in \mathfrak{k}^* : \eta \cdot A = 0 \ \forall \, A \in \mathfrak{k}_x\}$*.*

Proof. We first note that the tangent space $T_x(K \cdot x)$ to the orbit is the image of the infinitesimal action $\mathfrak{k} \to T_xX$ given by $A \mapsto A_{X,x}$ and the kernel of this map is $\mathfrak{k}_x$.

i) A tangent vector $\zeta \in T_xX$ is in the kernel of $d_x\mu$ if and only if for all $A \in \mathfrak{k}$ we have

$$0 = d_x\mu(\zeta) \cdot A = \omega_x(A_{X,x}, \zeta);$$

that is, if and only if $\zeta \in T_x(K \cdot x)^{\omega_x} := \{\zeta \in T_xX : \omega_x(\eta, \zeta) = 0 \, \forall \eta \in T_x(K \cdot x)\}$.

ii) If $\eta = d_x\mu(\zeta) \in \operatorname{Im} d_x\mu$ for some $\zeta \in T_xX$, then for all $A \in \mathfrak{k}$ we have that

$$\eta \cdot A = d_x\mu(\zeta) \cdot A = \omega_x(A_{X,x}, \zeta).$$

If $A \in \mathfrak{k}_x$, then $A_{X,x} = 0$ and, thus, $\eta = d_x\mu(\zeta) \in \operatorname{Ann}\mathfrak{k}_x$. Hence, we have an inclusion $\operatorname{Im} d_x\mu \subset \operatorname{Ann}\mathfrak{k}_x$. We will prove these vector spaces coincide, by checking that they have the same dimension. Let $d := \dim X$ and $n := \dim \ker d_x\mu$; then $\operatorname{Im} d_x\mu$ has dimension $d - n$. The short exact sequence

$$0 \to \operatorname{Ann}\mathfrak{k}_x \to \mathfrak{k}^* \to \mathfrak{k}_x^* \to 0$$

shows that

$$\dim \operatorname{Ann}\mathfrak{k}_x = \dim \mathfrak{k} - \dim \mathfrak{k}_x = \dim T_x(K \cdot X).$$

Therefore, it suffices to prove that $\dim T_x(K \cdot X) = d - n$. For any subspace $W \subset T_xX$, there is a short exact sequence

$$0 \to W^{\omega_x} \to T_xX \to W^* \to 0,$$

where the map $T_xX \to W^*$ is given by $\zeta \mapsto \omega_x(\zeta, -)|_W$. In particular, $\dim W + \dim W^{\omega_x} = \dim T_xX = d$. Applying this to $T_x(K \cdot x)$ and using Part i), we conclude that

$$\dim T_x(K \cdot x) = d - \dim(T_x(K \cdot x))^{\omega_x} = d - \dim \ker d_x\mu = d - n,$$

which completes the proof. □

We recall that the action of K on X is free if all stabilisers K_x are trivial. We say an action is locally free at x if the stabiliser group K_x is finite, which is if and only if $\mathfrak{k}_x = 0$.

Corollary 3.2. *Suppose we have a Hamiltonian action of a Lie group K on a symplectic manifold (X, ω) with associated moment map $\mu : X \to \mathfrak{k}^*$. Let η be an element in $\mathfrak{k}^*$ which is fixed by the coadjoint action of K. Then:*

i) *The action is locally free at $x \in X$ if and only if x is a regular point of μ.*
ii) *The K-action is locally free on $\mu^{-1}(\eta)$ if and only if η is a regular value of μ.*
iii) *If η is a regular value of μ, then $\mu^{-1}(\eta) \subset X$ is a closed submanifold of codimension equal to the dimension of $\mathfrak{k}$, which is preserved by the K-action. Furthermore, $T_x\mu^{-1}(\eta) = \ker d_x\mu$ for all $x \in \mu^{-1}(\eta)$, and the vector spaces $T_x\mu^{-1}(\eta)$ and $T_x(K \cdot x)$ are orthogonal with respect to the symplectic form ω_x on T_xX.*

Proof. i) The stabiliser K_x of a point x is finite if and only if its Lie algebra $\mathfrak{k}_x = 0$ is zero. By Lemma 3.1 ii), we have that $\operatorname{Im} d_x\mu = \operatorname{Ann}\mathfrak{k}_x$ which is equal to $\mathfrak{k}^*$ if and only if $\mathfrak{k}_x = 0$. Hence, the action is locally free at x if and only if $d_x\mu$ is surjective (that is, x is a regular point of μ). Then ii) follows from i) and the definition of regular value.

For iii), we use the preimage theorem for smooth manifolds: if $\mu : X \to \mathfrak{k}^*$ is a smooth map of smooth manifolds, then the preimage of a regular value is a closed submanifold of dimension $\dim X - \dim \mathfrak{k}^*$. Since $\eta \in \mathfrak{k}^*$ is fixed by the coadjoint action, equivariance of μ implies that the preimage $\mu^{-1}(\eta)$ is preserved by the action of K. As $\mu|_{\mu^{-1}(\eta)} = \eta$ is constant, $d_x\mu = 0$ on $T_x\mu^{-1}(\eta)$ for all $x \in \mu^{-1}(\eta)$. Hence, $T_x\mu^{-1}(\eta) \subset \ker d_x\mu$. Since η is a regular value, $d_x\mu$ is surjective and so

$$\dim \ker d_x\mu = \dim T_xX - \dim \mathfrak{k}^* = \dim X - \dim \mathfrak{k}^* = \dim \mu^{-1}(\eta);$$

thus $T_x\mu^{-1}(\eta) = \ker d_x\mu$. The final statement of iii) then follows from Lemma 3.1. □

3.2. **Symplectic reduction.** We suppose as above that we have a Hamiltonian action of a Lie group K on a symplectic manifold (X, ω) with associated moment map $\mu : X \to \mathfrak{k}^*$. We want to construct a quotient of the K-action on X (or a submanifold of X on which the action is free) in the symplectic category.

Definition 3.3. For a coadjoint fixed point $\eta \in \mathfrak{k}^*$, we define the *symplectic reduction* of the K-action on X at η to be the following topological quotient

$$X//_{\eta}^{\mathrm{red}}K := \mu^{-1}(\eta)/K.$$

This orbit space was considered by Marsden and Weinstein [4] and Meyer [6] as a possible symplectic quotient. At the moment, this quotient is just a topological space: its topology is the weakest topology for which the quotient map $\mu^{-1}(\eta) \to \mu^{-1}(\eta)/K$ is continuous. If η is a regular value of μ, then the preimage $\mu^{-1}(\eta)$ is a submanifold of X of dimension equal to $\dim X - \dim(K)$ and K acts on $\mu^{-1}(\eta)$ with finite stabilisers by Corollary 3.2. However, the action of K on $\mu^{-1}(\eta)$ may not be free and so the symplectic reduction will be an orbifold rather than a manifold. If η is a regular value and the action of K on $\mu^{-1}(\eta)$ is free and proper, then the symplectic reduction is a manifold of dimension $\dim X - 2\dim K$ by Theorem 1.3. In this case, there is a natural symplectic form on $\mu^{-1}(\eta)/K$; this is a theorem of Marsden and Weinstein [4] and Meyer [6] which we explain in the following section.

Remark 3.4. If η is a regular value of μ, but is not fixed by the coadjoint action, then we can instead consider the symplectic reduction

$$\mu^{-1}(\eta)/K_\eta$$

where $K_\eta = \{k \in K : \mathrm{Ad}_k^*\eta = \eta\}$ is the stabiliser group of η for the coadjoint action.

There is one point in the co-Lie algebra which is always fixed by the coadjoint action: the origin $0 \in \mathfrak{k}^*$. We refer to the symplectic reduction at zero simply as the symplectic reduction and write

$$X//^{\mathrm{red}}K := \mu^{-1}(0)/K.$$

We will later see that it provides a universal symplectic quotient of the K-action.

3.3. **Marsden-Weinstein-Meyer Theorem.**

Theorem 3.5. *Let K be a Lie group and suppose we have a Hamiltonian action of K on a symplectic manifold (X, ω) with moment map $\mu : X \to \mathfrak{k}^*$. Let $\eta \in \mathfrak{k}^*$ be coadjoint fixed. If the action of K on $\mu^{-1}(\eta)$ is free and proper, then the following statements hold.*

i) *The symplectic reduction $X//_{\eta}^{\mathrm{red}}K = \mu^{-1}(\eta)/K$ is a smooth manifold of dimension $\dim X - 2\dim K$. Furthermore, the quotient map $\pi : \mu^{-1}(\eta) \to \mu^{-1}(\eta)/K$ is a principal K-bundle.*

ii) *There is a unique symplectic form ω^{red} on X_{η}^{red} such that $\pi^*\omega^{\mathrm{red}} = i^*\omega$ where $i : \mu^{-1}(\eta) \hookrightarrow X$ denotes the inclusion and $\pi : \mu^{-1}(\eta) \to \mu^{-1}(\eta)/K$ is the quotient map.*

Remark 3.6.

(1) The assumption that the action of K on $\mu^{-1}(\eta)$ is free and proper is needed to prove that the symplectic reduction is a manifold. As the action of K on $\mu^{-1}(\eta)$ is free, it follows that η is a regular value of the moment map.

(2) If η is a regular value, then the action is locally free and so the topological quotient is at least an orbifold. If, moreover, K is compact, then the action on $\mu^{-1}(\eta)$ is proper and locally free. In the case, the symplectic reduction at η inherits an orbifold symplectic structure, by an orbifold version of the above theorem.

(3) More generally, there is a stratified symplectic structure on $\mu^{-1}(\eta)/K$ defined by Sjamaar and Lerman [7] such that each stratum is a smooth symplectic manifold and the symplectic reduction $\mu^{-1}(\eta)/K$ has a Poisson structure for which the strata are symplectic leaves. The stratification is obtained by stratifying $\mu^{-1}(\eta)$ by the conjugacy class of the stabiliser group for the action.

Before we prove the above theorem, we need a few preliminary lemmas.

Lemma 3.7. *With the assumptions of the above theorem, for all $x \in \mu^{-1}(\eta)$, the subspace $T_x(K \cdot x)$ of T_xX is an isotropic subspace.*

Proof. We recall that $T_x(K\cdot x)$ is an isotropic subspace of the symplectic vector space (T_xX, ω_x) if $T_x(K\cdot x) \subset T_x(K\cdot x)^{\omega_x}$. The subspaces $\ker d_x\mu = T_x\mu^{-1}(\eta)$ and $T_x(K\cdot x)$ of T_xX are symplectic orthogonal complements with respect to ω_x for $x \in \mu^{-1}(\eta)$ by Corollary 3.2. As η is fixed by the coadjoint action, this implies $\mu^{-1}(\eta)$ is K-invariant and so $K \cdot x \subset \mu^{-1}(\eta)$. Therefore

$$T_x(K \cdot x) \subset T_x\mu^{-1}(\eta) = T_x(K \cdot x)^{\omega_x},$$

which completes the proof that $T_x(K \cdot x)$ is an isotropic subspace of (T_xX, ω_x). □

Lemma 3.8. *Let I be an isotropic subspace of a symplectic vector space (V, ω). Then ω induces a unique symplectic form ω' on the quotient I^ω/I.*

Proof. We define

$$\omega'([v], [w]) := \omega(v, w)$$

and check this definition is well defined:

$$\begin{aligned}\omega'(v + i, w + j) &= \omega(v, w) + \omega(i, w) + \omega(v, j) + \omega(i, j)\\ &= \omega(v, w) + 0 + 0 + 0\end{aligned}$$

for $i, j \in I$. The non-degeneracy of ω' follows from that of ω: if $[u] \in I^\omega/I$ and $\omega'([u], [v]) = 0$ for all $v \in I^\omega/I$, then $\omega(u, v) = 0$ for all $v \in I^\omega$ and so $u \in (I^\omega)^\omega = I$ i.e. $[u] = 0$. □

Proof. (Marsden-Weinstein-Meyer Theorem) The preimage theorem shows that $\mu^{-1}(\eta)$ is a closed smooth submanifold of X of dimension $\dim X - \dim K$. Furthermore, as K acts on $\mu^{-1}(\eta)$ freely and properly, the quotient $Y := \mu^{-1}(\eta)/K$ is a smooth manifold of dimension $\dim X - 2\dim K$. We shall construct a non-degenerate 2-form ω^{red} on Y such that $\pi^*\omega^{\text{red}} = i^*\omega$, by constructing symmetric forms ω_p^{red} on T_pY for all $p \in Y$. Let $p = \pi(x)$ where $\pi : \mu^{-1}(\eta) \to Y = \mu^{-1}(\eta)/K$; then we have a short exact sequence of vector spaces

$$0 \to T_x(K \cdot x) \to T_x\mu^{-1}(\eta) \to T_pY \to 0.$$

By Lemma 3.7, the subspace $T_x(K \cdot x)$ is isotropic and has symplectic orthogonal complement $T_x(\mu^{-1}(\eta))$. By Lemma 3.8, there is a canonical symplectic form ω_p^{red} on

$$T_x(K \cdot x)^{\omega_x}/T_x(K \cdot x) = T_x\mu^{-1}(\eta)/T_x(K \cdot x) \cong T_pY.$$

By construction, this is a non-degenerate 2-form such that $\pi^*\omega^{\text{red}} = i^*\omega$, and so it remains to check that this symplectic form is closed. As the exterior derivative d commutes with pullback we have that

$$\pi^* d\omega^{\text{red}} = d\pi^*\omega^{\text{red}} = di^*\omega = i^*d\omega = 0.$$

The pullback map $\pi^* : \Omega^3(X_\eta^{\text{red}}) \to \Omega^3(\mu^{-1}(\eta))$ is injective, as π is surjective, and so we conclude that $d\omega^{\text{red}} = 0$. □

Example 3.9. Consider the action of $\text{U}(1) \cong S^1$ on $\mathbb{C}^n$ by multiplication $s \cdot (a_1, \dots, a_n) = (sa_1, \dots, sa_n)$. We can take the standard symplectic form on $\mathbb{C}^n$ and use the Killing form on $\mathfrak{u}(1)$ to identify $\mathfrak{u}(1)^* \cong \mathfrak{u}(1) \cong \mathbb{R}$ and write the moment map for this action as

$$\mu(x_1, \dots, x_n) = \sum_{k=1}^{n} |x_k|^2.$$

For $\eta = 0$, we have that $\mu^{-1}(0) = \{0\}$ and $\mu^{-1}(0)/K$ is just a point. The value $\eta = 1$ is more interesting:

$$\mu^{-1}(1) = S^{2n-1} = \{(x_1, \ldots, x_n) : \sum |x_k|^2 = 1\}$$

and the symplectic reduction is $\mathbb{C}^n//^{\mathrm{red}}S^1 = \mu^{-1}(1)/S^1 = S^{2n-1}/S^1 = \mathbb{P}^{n-1}$. The symplectic form ω induced from the standard symplectic form on $\mathbb{C}^n$ is the Fubini–Study form. In fact, if one considers the construction of the Fubini–Study form using the standard form on $\mathbb{C}^{n+1}$, then one can now see that this is just a special case of the Marsden–Weinstein–Meyer Theorem.

Example 3.10. Consider the action of $K = \mathrm{U}(m)$ on the space of $l \times m$-matrices over the complex numbers $M_{l\times m}(\mathbb{C}) \cong \mathbb{C}^{lm}$ as in Example 2.11 where $l > m$. We recall that the moment map is given by

$$\mu(M) \cdot A = \frac{i}{2}\mathrm{Tr}(MAM^*)$$

for $M \in M_{l\times m}$ and $A \in \mathfrak{u}(m)$. By using the Killing form on $\mathfrak{u}(m)$, we can identify $\mathfrak{u}(m)^* \cong \mathfrak{u}(m)$ and view the moment map as a morphism $\mu : M_{l\times m} \to \mathfrak{u}(m)$ given by

$$\mu(M) = \frac{i}{2}M^*M.$$

Let $\eta = iI_m/2$ denote the skew-Hermitian matrix which is an (imaginary) scalar multiple of the identity matrix I_m; then clearly η is fixed by the adjoint action of $\mathrm{U}(m)$ on $\mathfrak{u}(m)$. The preimage $\mu^{-1}(\eta) = \{M \in M_{l\times m} : M^*M = I_m\}$ consists of $l \times m$ matrices whose m columns are linearly independent and define a length m unitary frame of $\mathbb{C}^l$. The symplectic reduction $\mu^{-1}(\eta)/\mathrm{U}(m)$ is the grassmannian $\mathrm{Gr}(m, l)$ of m-planes in $\mathbb{C}^l$.

There is a more general version of the Marsden-Weinstein-Meyer Theorem which allows us to take reductions at points which are not fixed by the coadjoint action:

Proposition 3.11. *Given a Hamiltonian action of a compact connected Lie group K on a symplectic manifold (X, ω) with moment map $\mu : X \to \mathfrak{k}^*$ and an orbit $\mathcal{O}$ for the coadjoint action of K on $\mathfrak{k}^*$. If the orbit consists of regular values of μ and the action of K on $\mu^{-1}(\mathcal{O})$ is free and proper, then the symplectic reduction $\mu^{-1}(\mathcal{O})/K$ is a symplectic manifold of dimension $\dim X + \dim \mathcal{O} - 2\dim K$.*

Proof. The assumption that every point of $\mathcal{O}$ is a regular value of the moment map means that the preimage $\mu^{-1}(\mathcal{O})$ is a closed submanifold of X of dimension $\dim X + \dim \mathcal{O} - \dim K$. The coadjoint orbit $\mathcal{O}$ has a natural symplectic form, which we denote by $\omega_{\mathcal{O}}$, given by the Kostant-Kirillov symplectic structure. Consider the natural action of K on the product $(X \times \mathcal{O}, -\omega \boxplus \omega_{\mathcal{O}})$, for which the moment map $\mu' : X \times \mathcal{O} \to \mathfrak{k}^*$ is given by

$$\mu'(x, \eta) = -\mu(x) + \eta.$$

Then the proposition follows by applying the original version of the Marsden-Weinstein-Meyer Theorem to the regular value 0 of μ' and observing that $\mu^{-1}(\mathcal{O}) \cong (\mu')^{-1}(0)$. □

Remark 3.12. If $\eta \in \mathfrak{k}^*$ is not fixed by the coadjoint action of K on $\mathfrak{k}^*$, then

$$K_\eta = \{k \in K : Ad_k^*\eta = \eta\}$$

acts on $\mu^{-1}(\eta)$. Then the symplectic reduction $\mu^{-1}(\mathcal{O}_\eta)/K$ constructed above for the coadjoint orbit $\mathcal{O}_\eta$ of η is homeomorphic to the quotient $\mu^{-1}(\eta)/K_\eta$.

Remark 3.13. Suppose X is a Kähler manifold (i.e. it has a complex structure I and a Riemannian metric g such that $\omega := g(I-, -)$ is a symplectic form) and the action of the Lie group K preserves this structure (since the metric g, complex structure I and symplectic form ω are compatible, it suffices to check that K preserves three out of the two structures). If the action is Hamiltonian with moment map μ and K acts freely and properly on $\mu^{-1}(0)$ where 0 is a regular value of the moment map, then the symplectic reduction $\mu^{-1}(0)/X$ also has a Kähler structure (i.e. the almost complex structure induced on the symplectic reduction is integrable and and the metric also induces a compatible kähler metric on the symplectic reduction).

3.4. **Symplectic Implosion.** Let K be a compact Lie group with a Hamiltonian action on a symplectic manifold (X, ω) with moment map $\mu : X \to \mathfrak{k}^*$. The idea of a symplectic implosion for this action is to construct a stratified symplectic space with an action of a maximal torus of K, such that the symplectic reductions of K acting on X at different values η can be recovered as symplectic reductions of this maximal torus acting on the symplectic implosion. In this sense, it provides an abelianisation of the process of symplectic reduction.

Definition 3.14. Fix a maximal torus $T \subset K$ and positive Weyl chamber $\mathfrak{t}^*_+$. The *symplectic implosion* X_{impl} of X with respect to $\mathfrak{t}^*_+$ is a stratified symplectic space with a Hamiltonian action of T on X_{impl} and moment map $\mu_{\text{impl}} : X_{\text{impl}} \to \mathfrak{t}^*$ such that for each $\eta \in \overline{\mathfrak{t}^*_+}$, we have a (stratified) symplectomorphism

$$\mu_{\text{impl}}^{-1}(\eta)/T \cong \mu^{-1}(\eta)/K_\eta.$$

The construction of symplectic implosions is due to Guillemin, Jeffrey and Sjamaar [3]. The strata of X_{impl} are indexed by the faces σ of $\mathfrak{t}^*_+$. More precisely, we have

$$X_{\text{impl}} = \bigsqcup_\sigma \mu^{-1}(\sigma)/[K_\sigma, K_\sigma],$$

where σ runs over the faces of $\mathfrak{t}^*_+$, and K_σ denotes the stabiliser of the face σ.

Alternatively, one can construct X_{impl} by gluing points in $\mu^{-1}(\mathfrak{t}^*_+)$, where $x \sim y$ if and only if $\mu(x) = \mu(y) = \eta$ and $x = ky$ for some $k \in [K_\eta, K_\eta]$. From this second description, we see there is a T-action on $\mu^{-1}(\mathfrak{t}^*_+)$, which descends to $X_{\text{impl}} = \mu^{-1}(\mathfrak{t}^*_+)/\sim$, as T normalises $[K_\eta, K_\eta]$. The moment map $\mu : X \to \mathfrak{k}^*$ induces the moment map $\mu_{\text{impl}} : X_{\text{impl}} \to \mathfrak{t}^*$, in the sense that $\mu_{\text{impl}} \circ \pi = \mu|_{\mu^{-1}(\mathfrak{t}^*_+)}$, where $\pi : \mu^{-1}(\mathfrak{t}^*_+) \to X_{\text{impl}}$ is the quotient map obtained by gluing.

Theorem 3.15. *(Guillemin, Jeffrey, Sjamaar* [3]*) The stratified symplectic space X_{impl} is a symplectic implosion of the K-action on X with respect to $\mathfrak{t}^*_+$.*

A third method for constructing the symplectic implosion is to use the universal symplectic implosion for K: namely consider K acting on its cotangent bundle T^*K; then Guillemin, Jeffrey and Sjamaar [3] prove there is an isomorphism of stratified Hamiltonian T-spaces:

$$X_{\text{impl}} \cong (X \times (T^*K)_{\text{impl}})/\!/^{\text{red}} K.$$

3.5. **Lagrangian Correspondences.** We recall that a symplectomorphism $f : (M, \omega) \to (M', \omega')$ of symplectic manifolds is a diffeomorphism $f : M \to M'$ such that $f^*\omega' = \omega$. Since this definition is rather restrictive, one has a more general notion of morphisms in the symplectic category given by Lagrangian correspondences.

Definition 3.16.

(1) A submanifold L of a symplectic manifold X is *Lagrangian* if $2 \dim L = \dim X$ and $i^*\omega = 0$ where $i : L \hookrightarrow X$ is the inclusion. Equivalently, L is Lagrangian if for all $x \in L$ the vector space T_xL is a Lagrangian subspace of T_xX; that is,

$$(T_xL)^{\omega_x} = \{\eta \in T_xX : \omega_x(\eta, \zeta) = 0 \,\forall\, \zeta \in T_xL\} = T_xL.$$

(2) A *Lagrangian correspondence* between symplectic manifolds (X_1, ω_1) and (X_2, ω_2) is a Lagrangian submanifold L_{12} of $(X_1 \times X_2, -\omega_1 \boxplus \omega_2)$ where $-\omega_1 \boxplus \omega_2 := -\pi_1^*\omega_1 + \pi_2^*\omega$.

Remark 3.17. For any symplectomorphism $\phi : (X_1, \omega_1) \to (X_2, \omega_2)$, the graph $\Gamma(\phi) \subset (X_1 \times X_2, -\omega_1 \boxplus \omega_2)$ is a Lagrangian submanifold. Therefore, the notion of symplectic correspondence generalises that of symplectomorphisms.

We want to view Lagrangian correspondences as morphisms in the symplectic category. For this, we need to define the composition of two Lagrangian correspondence. Given Lagrangian submanifolds $L_{12} \subset (X_1 \times X_2, -\omega_1 \boxplus \omega_2)$ and $L_{23} \subset (X_2 \times X_3, -\omega_2 \boxplus \omega_3)$, we define the composition $L_{13} = L_{23} \circ L_{12}$ to be the Lagrangian submanifold $L_{13} := \pi_{13}(L_{12} \times_{X_2} L_{23})$ of $(X_1 \times X_3, -\omega_1 \boxplus \omega_3)$ given by

$$L_{13} = \{(x_1, x_3) \in X_1 \times X_3 : \exists x_2 \in X_2 \text{ such that } (x_1, x_2) \in L_{12} \text{ and } (x_2, x_3) \in L_{23}\}.$$

Then, following [8], we define morphisms in the symplectic category to consist of chains of symplectic correspondences.

3.6. **Universality of the symplectic reduction.** Given a Hamiltonian action of a Lie group K on a symplectic manifold (X, ω) with moment map $\mu : X \to \mathfrak{k}^*$, the aim of this section is to show (under the assumptions of the Marsden-Weinstein-Meyer theorem) that the symplectic reduction $(\mu^{-1}(0)/K, \omega^{\text{red}})$ is a universal quotient of the K-action on (X, ω) in the symplectic category. We recall that the symplectic form ω^{red} is the unique symplectic form such that $\pi^*\omega^{\text{red}} = i^*\omega$ where $i : \mu^{-1}(0) \hookrightarrow X$ denotes the inclusion and $\pi : \mu^{-1}(0) \to \mu^{-1}(0)/K$ denotes the quotient map. First of all we need to construct a Lagrangian correspondence between (X, ω) and the symplectic reduction $(\mu^{-1}(0)/K, \omega^{\text{red}})$.

Lemma 3.18. *If the assumptions of the Marsden-Weinstein-Meyer Theorem hold for $\eta = 0$, then*

$$L_\mu := \text{Im}\left(i \times \pi : \mu^{-1}(0) \to X \times \mu^{-1}(0)/K\right)$$

is a Lagrangian submanifold of $(X \times \mu^{-1}(0)/K, -\omega \boxplus \omega^{red})$.

Proof. As L_μ is diffeomorphic to $\mu^{-1}(0)$ we have

$$\dim L_\mu = \dim X - \dim K = \frac{1}{2} \dim\left(X \times \mu^{-1}(0)/K\right).$$

If $j : L_\mu \hookrightarrow X \times \mu^{-1}(0)/K$, then $j^*(-\omega \boxplus \omega^{\text{red}}) \equiv 0$ if and only if $-i^*\omega + \pi^*\omega^{\text{red}} \equiv 0$ which holds by the Marsden-Weinstein-Meyer theorem. □

In particular, this lemma gives a morphism in the symplectic category (X, ω) and the reduction $(\mu^{-1}(0)/K, \omega^{\text{red}})$. However, we want this morphism to be K-equivariant and so we should define what it means for a Lagrangian correspondence to be K-invariant (or in fact more generally K-equivariant):

Definition 3.19.

(1) A *K-equivariant Lagrangian correspondence* between symplectic manifolds (x_i, ω_i) for $i = 1, 2$ with Hamiltonian K-actions is a Lagrangian submanifold $L \subset (X_1 \times X_2, -\omega_1 \boxplus \omega_2)$ which is K-invariant and satisfies $\mu_{12}(L) = 0$, where μ_{12} denotes the moment map for the K-action on $(X_1 \times X_2, -\omega_1 \boxplus \omega_2)$.

(2) A *K-invariant Lagrangian correspondence* between a symplectic manifold (X_1, ω_1) with Hamiltonian K-action and a symplectic manifold (X_2, ω_2) is a K-equivariant Lagrangian correspondence, where we give (X_2, ω_2) the trivial K-action.

The proof of the following lemma is immediate by definition of L_μ.

Lemma 3.20. *If the assumptions of the Marsden-Weinstein-Meyer Theorem hold for $\eta = 0$, then the Lagrangian correspondence L_μ is K-invariant.*

Proposition 3.21. *Let K be a Lie group with a Hamiltonian action on a symplectic manifold (X, ω) with moment map $\mu : X \to \mathfrak{k}^*$. If K acts freely and properly on $\mu^{-1}(0)$, then L_μ is a universal K-invariant symplectic correspondence from (X, ω); in the sense that every other K-invariant Lagrangian correspondence from (X, ω) to a symplectic manifold (Y, ω') factors through L_μ.*

Proof. It suffices to prove the result when the morphism from (X, ω) to (Y, ω') is given by a single K-invariant Lagrangian correspondence i.e. a Lagrangian submanifold $L' \subset (X \times Y, -\omega \boxplus \omega')$ which is preserved by the action of K and on which $\mu_{XY} : X \times Y \to \mathfrak{k}^*$ is zero. As the K-action on Y is trivial, so is the moment map μ_Y and so μ_{XY} is the projection $X \times Y \to X$ followed by $\mu = \mu_X : X \to \mathfrak{k}^*$. In particular, $L' \subset \mu^{-1}(0) \times Y$ as $\mu_{XY}(L') = 0$, by assumption that L' is K-invariant. To show that L' factors through L_μ, it suffices to produce a Lagrangian correspondence L'' between $\mu^{-1}(0)/K$ and Y such that $L' = L'' \circ L_\mu$. One checks that $L'' := L'/K \subset \mu^{-1}(0)/K \times Y$ is the required Lagrangian submanifold. □

References

1. J. Butterfield, *On symplectic reduction in classical mechanics*, Handbook of philosophy of physics, Elsevier Oxford, UK, 2007, pp. 1–132.
2. A. Cannas da Silva, *Lecture on symplectic geometry*, Lecture notes in Mathematics, Springer-Verlag, 2001.
3. V. Guillemin, L. Jeffrey, and R. Sjamaar, *Symplectic implosion*, Transform. Groups **7** (2002), no. 2, 155–184. MR 1903116
4. J. Marsden and A. Weinstein, *Reduction of symplectic manifolds with symmetry*, Rep. Math. Phys. **5** (1974), 121–130.
5. D. McDuff and D. Salamon, *Introduction to symplectic topology*, second ed., Oxford Mathematical Monographs, Oxford University Press, 1998.
6. K. Meyer, *Symmetries and integrals in mechanics*, Dynamical systems (M. Peixoto, ed.), Academic Press New York, 1973, pp. 259–273.
7. R. Sjamaar and E. Lerman, *Stratified symplectic spaces and reduction*, Ann. of Math. **134** (1991), no. 2, 375–422.
8. A. Weinstein, *The symplectic "category"*, Differential geometric methods in mathematical physics (Clausthal, 1980), Lecture Notes in Mathematics, vol. 905, Springer, Berlin, 1982, pp. 45–51.
9. C. Woodward, *Moment maps and geometric invariant theory*, Les cours du CIRM **1** (2010), no. 1, 55–98.